Programming Languages and Computation

Week 8: Loop Invariants

** 1. For each of the following statements, determine whether $q \le x$ is a loop invariant. If so, justify why. Otherwise, provide a counter-example.

(a) while $y * q \le x$ do $q \leftarrow q + 1$

(b) while $!(q \le x)$ do $x \leftarrow x - 1$

(c) while $!(x \le q * q)$ do $q \leftarrow q + 1$

** 2. Find a loop invariant that can be used to demonstrate the following Hoare triple. You should justify your solution in relation to the rules of Hoare logic or the strongest post-condition.

$$
\begin{array}{l}
\{0 \le x \ \&\& \ 0 < y\} \\
q \leftarrow 0; \\
r \leftarrow x; \\
\text{while } y \le r \text{ do} \\
\quad r \leftarrow r - y; \\
\quad q \leftarrow q + 1; \\
\{x = q * y + r \ \&\& \ r < y\}
\end{array}
$$

*** 3. Find a loop invariant that can be used to demonstrate the following Hoare triple. You should justify your solution in relation to the rules of Hoare logic or the strongest post-condition.

$$
\begin{array}{l}
\{n \ge 0\} \\
i \leftarrow 0 \\
s \leftarrow 0 \\
\text{while } !(i = n) \text{ do} \\
\quad s \leftarrow s + 2 * i + 1 \\
\quad i \leftarrow i + 1 \\
\{s = n * n\}
\end{array}
$$

*** 4. Find a pair of loop invariants (one for each loop) that together can be used to demonstrate the following Hoare triple. You should justify your solution in relation to the rules of Hoare logic or the strongest post-condition.

$$
\begin{array}{l}
\{0 \leq n\} \\
i \leftarrow 0; \\
j \leftarrow 0; \\
\textsf{while } i \leq n-1 \textsf{ do} \\
\quad j \leftarrow 0; \\
\quad \textsf{while } j \leq i \textsf{ do} \\
\quad\quad j \leftarrow j+1; \\
\quad i \leftarrow i+1 \\
\{j = n\}
\end{array}
$$

** 5. Find a loop invariant for the following program that demonstrates that this program has non-terminating traces. You should justify your answer in relation to the definition of a Hoare triple.

$$\textsf{while } x > 0 \textsf{ do } x \leftarrow x + y$$