Part II: Semantics

Eddie Jones MUB 2.60

- Giving meaning to programs
  - Definition
  - Denotational
  - Operational
  - Axiomatic
  - W hile Programs
  - Abstract Syntax Trees

Simple Arithmetic Expressions

$S \to n \mid S \pm S \mid S - S \mid S * S$

$n \in \mathbb{Z}$

$1 \pm (2 \pm 3)$

$1 + 1 = 2$

$1 \pm 1 \neq 1$

$= [\![1]\!]_S + [\![\ 2 \ \ 3\ ]\!]_S$

$= 1 + ([\![2]\!]_S + [\![3]\!]_S)$

$= 1 + (2 + 3)$

$= 6$

$[\![\cdot]\!]_S : S \to \mathbb{Z}$

Denotation function

$[\![n]\!]_S = n$

$[\![e_1 \pm e_2]\!]_S = [\![e_1]\!]_S + [\![e_2]\!]_S$

$[\![e_1 - e_2]\!]_S = [\![e_1]\!]_S - [\![e_2]\!]_S$

$[\![e_1 * e_2]\!]_S = [\![e_1]\!]_S \cdot [\![e_2]\!]_S$

Arithmetic Expressions

$A \to x \mid n \mid A + A \mid A - A \mid A * A$

$x \in Var$

Context dependent Value

Program State

$State = Var \to \mathbb{Z}$ $\quad \sigma, \sigma_1, \sigma_2, \ldots$ to be states

$[x \mapsto 1, y \mapsto 2] \in State$.

"$z \mapsto 0$"...

$[\![\cdot]\!]_A : A \to (State \to \mathbb{Z})$

$\cong A \times State \to \mathbb{Z}$

$[\![x]\!]_A(\sigma) = \sigma(x) \in \mathbb{Z}$

$[\![n]\!]_A(\sigma) = n$

$[\![e_1 + e_2]\!](\sigma) = [\![e_1]\!]_A(\sigma) + [\![e_2]\!]_A(\sigma)$ recursions pass state.

$[\![e_1 - e_2]\!](\sigma) = [\![e_1]\!]_A(\sigma) - [\![e_2]\!]_A(\sigma)$

$[\![e_1 * e_2]\!](\sigma) = [\![e_1]\!]_A(\sigma) \cdot [\![e_2]\!]_A(\sigma)$

$([x \mapsto -1])$

$= [\![x]\!]_A([x \mapsto -1]) + [\![1]\!]_A([x \mapsto -1])$

$= (-1) + 1$

$= 0$

Boolean Expressions

$B \to true \mid false \mid !B$

$\mid B \,\&\&\, B \mid B \parallel B$

$\mid A = A \mid A \leq A$

$[\![\cdot]\!]_B : B \to (State \to \{\bot, \top\})$

$\cong B \times State \to \{\bot, \top\}$

$[\![true]\!]_B(\sigma) = \top$

$[\![false]\!]_B(\sigma) = \bot$

$[\![!b]\!]_B(\sigma) = \neg [\![b]\!]_B(\sigma)$

$[\![b_1 \,\&\&\, b_2]\!]_B(\sigma) = [\![b_1]\!]_B(\sigma) \wedge [\![b_2]\!](\sigma)$

$[\![b_1 \parallel b_2]\!]_B(\sigma) = [\![b_1]\!]_B(\sigma) \vee [\![b_2]\!]_B(\sigma)$

$[\![a_1 = a_2]\!]_B(\sigma) = [\![a_1]\!]_A(\sigma) \equiv [\![a_2]\!]_A(\sigma)$

$[\![a_1 \leq a_2]\!]_B(\sigma) = [\![a_1]\!]_A(\sigma) \leq [\![a_2]\!]_A(\sigma)$

Semantically equivalent

$[\![e_1]\!]_A = [\![e_2]\!]_A \in State \to \mathbb{Z}$

$\iff \forall \sigma \in State.\ [\![e_1]\!]_A(\sigma) = [\![e_2]\!]_A(\sigma) \in \mathbb{Z}$

$[\![x + x]\!]_A = [\![x * 2]\!]_A$

Proof.

Let $\sigma \in State$.

$[\![x + x]\!]_A(\sigma) = [\![x]\!]_A(\sigma) + [\![x]\!]_A(\sigma)$

$= \sigma(x) + \sigma(x)$

$= 2 \cdot \sigma(x)$

$[\![x * 2]\!]_A(\sigma) = \cdots = 2 \cdot \sigma(x)$

$\therefore$ $x + x$ is semantically equivalent to $x * 2$